APPLICATION OF THE SECOND METHOD OF LIAPUNOV IN THE INVESTIGATION OF THE STEADY MOTION OF A GYROSCOPE WHEN ELASTIC PROPERTIES OF THE ROTOR AXIS ARE TAKEN INTO ACCOUNT

(PRIMENENIE VTOROGO METODA LIAPUNOVA K IZUCHENIIU USTOICHIVOSTI STATSIONARNYKH DVIZHENII GIROSKOPA S UCHETOM UPRUGIKH SVOISTV OSI ROTORA)

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We shall consider the rotor of a gyroscope as a heavy uniform fly-wheel placed symmetrically on a slender weightless axis with ends fixed in the inner ring. This mechanical system has, in general, an infinite number of degrees of freedom, and for such a case the Liapunov method has not been worked out; consequently, its stability must be investigated through approximate methods. In many cases a system with an infinite number of degrees of freedom can be approximated by a model with a finite number of degrees of freedom having basic mechanical properties of the original system, but being much easier to investigate. For example, when Chetaev [1] studied the stability of steady motions of a fly-wheel fixed on a stationary, slender, vertical shaft he approximated his original system by a model with three degrees of freedom. In this model the fly-wheel moves only in a horizontal plane, two coordinates determine the center of gravity of the fly-wheel in a fixed horizontal plane, the third coordinate is the rotation angle of the fly-wheel.

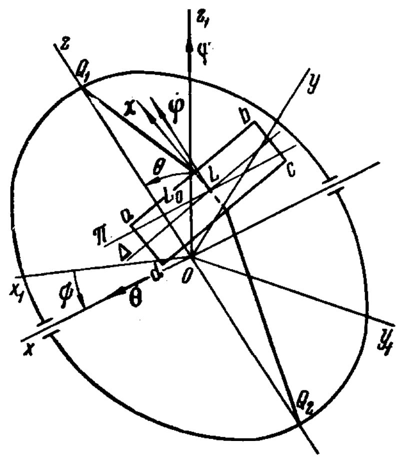

Fig. 1.

The model of our problem is shown in Fig. 1.

Following the conventional notation, ψ is the rotation angle of the outer ring about its vertical axis z_1, θ is the rotation angle of the inner ring (casing) about the x-axis, *abcd* is the rotor on the elastic axis Q_1LS_2 which passes through the center of gravity of the rotor L (sideview), zQ_1OQ_2 is the line in the plane of the inner ring joining its center (the fixed point O) with the points Q_1 and Q_2 (points where the rotor axis meets the inner ring). When the rotor axis is not bent then it coincides with the line Q_1OQ_2 and the point L coincides with the point L_0 whose distance from the fixed point O is ζ. The letter π denotes the plane through L_0 perpendicular to OQ_1. We assume that the center of gravity of the rotor remains all the time in the plane π, and its position is determined by the polar coordinates $r = L_0L$ and ϕ, where ϕ is the angle between the radius vector L_0L and the axis x^0. The axis x^0 is the intersection of the π-plane and the plane of the inner ring (the x^0-axis and the x-axis are parallel). The xyz-coordinate system is fixed in the inner ring, the $x^*y^*z^*$-coordinate system is parallel to the xyz-system and has its origin at the point L; the axes x', y' are in the plane of the rotor's central cross-section, denoted by Δ. The x'-axis is the line of intersection of the π-plane and the plane of the rotor's central cross-section and is assumed to be perpendicular to L_0L. The orientation of the rotor with respect to the $x^*y^*z^*$-system is determined by the angles δ (inclination with respect to the plane π), and χ (rotation angle of the rotor). Cosines of the angles between the x^*-, y^*-, z^*-, and x'-, y'-, z'-axes are given in the table.

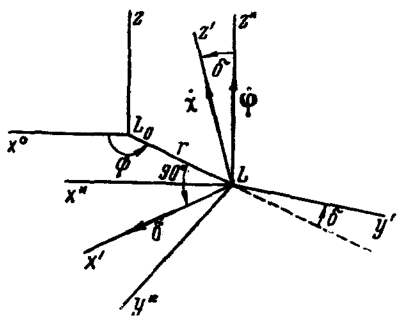

Fig. 2.

	x^*	y^*	z^*
x'	$\sin\varphi$	$-\cos\varphi$	0
y'	$\cos\varphi\cos\delta$	$\sin\varphi\cos\delta$	$\sin\delta$
z'	$-\cos\varphi\sin\delta$	$-\sin\varphi\sin\delta$	$\cos\delta$

The elastic properties of the rotor's axis are characterized through the restoring force $m\mu_1 r$, and the elastic moment $m\mu_2\delta$, where m is the mass of the rotor, μ_1 and μ_2 are the positive coefficients of rigidity of the axis.

The coordinates of the point L in the xyz-system are $x = r\cos\phi$, $y = r\sin\phi$, $z = \zeta$, and in the $x_1y_1z_1$-system are

$$x_1 = r\cos\varphi\cos\psi - r\sin\varphi\cos\theta\sin\psi + \zeta\sin\theta\sin\psi$$
$$y_1 = r\cos\varphi\sin\psi + r\sin\varphi\cos\theta\cos\psi - \zeta\sin\theta\cos\psi$$
$$z_1 = r\sin\theta\sin\varphi + \zeta\cos\theta$$

The force function of our mechanical system is

$$2U = -m\mu_1 r^2 - m\mu_2 \delta^2 - 2mg\,(r\sin\theta\sin\varphi + \zeta\cos\theta)$$

Let I be the moment of inertia of the outer ring with respect to the z_1-axis. Then the kinetic energy of the outer ring, of the casing, and of the gyroscope equal, respectively

$$2T^{(\mathrm{BH})} = I\dot\psi^2$$
$$2T^{(\mathrm{K})} = A_1\dot\theta^2 + B_1\dot\psi^2\sin^2\theta + C_1\dot\psi^2\cos^2\theta$$
$$2T^{(\Gamma)} = m\,(\dot x_1^2 + \dot y_1^2 + \dot z_1^2) + A\,(\Omega_{x'}^2 + \Omega_{y'}^2) + C\Omega_{z'}^2$$

Here A_1, B_1, C_1 are the moments of inertia of the casing about the x'-, y'-, z'-axis, respectively, $\Omega_{x'}$, $\Omega_{y'}$, $\Omega_{z'}$, are the x'-, y'-, z'-components of the rotor's instantaneous angular velocity vector

$$\Omega = \dot\psi + \dot\theta + \dot\chi + \dot\varphi + \dot\delta$$

It can be easily shown that these components equal

$$\Omega_{x'} = -\dot\psi\sin\theta\cos\varphi + \dot\theta\sin\varphi + \dot\delta$$
$$\Omega_{y'} = \dot\psi\,(\sin\theta\sin\varphi\cos\delta + \cos\theta\sin\delta) + \dot\theta\cos\varphi\cos\delta + \dot\varphi\sin\delta$$
$$\Omega_{z'} = -\dot\psi\,(\sin\theta\sin\varphi\sin\delta - \cos\theta\cos\delta) - \dot\theta\cos\varphi\sin\delta + \dot\varphi\cos\delta + \dot\chi$$

The kinetic energy of the whole system can be expressed in the form

$$\begin{aligned}2T = m\,[\dot r^2 + r^2\dot\varphi^2 + r^2\dot\psi^2(\cos^2\varphi + \sin^2\varphi\cos^2\theta) - r\zeta\dot\psi^2\sin 2\theta\sin\varphi +\\ + r^2\dot\theta^2\sin^2\varphi + \zeta^2(\dot\theta^2 + \dot\psi^2\sin^2\theta) - 2\zeta\dot r\dot\theta\sin\varphi + 2\zeta\dot r\dot\psi\sin\theta\cos\varphi +\\ + 2r\,(r\cos\theta - \zeta\sin\theta\sin\varphi)\,\dot\varphi\dot\psi - 2r\zeta\dot\varphi\dot\theta\cos\varphi - 2r\cos\varphi\,(r\sin\theta\sin\varphi + \zeta\cos\theta)\,\dot\theta\dot\psi +\\ + A_1\dot\theta^2 + [B_1\sin^2\theta + C_1\cos^2\theta + I]\,\dot\psi^2 + A\,(\Omega_{x'}^2 + \Omega_{y'}^2) + C\Omega_{z'}^2]\end{aligned}$$

Since the variables ψ, θ, χ, r, ϕ, δ are independent and holonomic, we can write the equations of motion of our system in the Lagrange form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i} \qquad (i = 1, \ldots, 6) \tag{1}$$
$$(q_1 = \psi,\ q_2 = \theta,\ q_3 = \chi,\ q_4 = r,\ q_5 = \varphi,\quad q_6 = \delta)$$

Equations (1) permit three first integrals: the integral of conservation of energy

$$T - U = \mathrm{const} \tag{2}$$

and two cyclic integrals with respect to the coordinates ψ, χ

$$\partial T/\partial\dot\psi = m\,[r^2\dot\psi(\cos^2\varphi+\sin^2\varphi\cos^2\theta)-r\zeta\dot\psi\sin 2\theta\sin\varphi+\zeta^2\dot\psi\sin^2\theta+$$
$$+\zeta\dot r\sin\theta\cos\varphi+r(r\cos\theta-\zeta\sin\theta\sin\varphi)\dot\varphi-r\cos\varphi(r\sin\theta\sin\varphi+\zeta\cos\theta)\dot\theta]+ \quad (3)$$
$$+(B_1\sin^2\theta+C_1\cos^2\theta+I)\dot\psi-A\Omega_{x'}\sin\theta\cos\varphi+A\Omega_{y'}(\sin\theta\sin\varphi\cos\delta+$$
$$\cos\theta\sin\delta)++C\Omega_{z'}(\cos\theta\cos\delta-\sin\theta\sin\varphi\sin\delta)=\text{const}$$
$$\Omega_{z'}=\text{const} \quad (4)$$

Equations (1) have stationary solutions

$$\theta=\theta_0,\quad r=r_0,\quad \varphi=\varphi_0,\quad \delta=\delta_0,\quad \dot\theta=\dot r=\dot\varphi=\dot\delta=0,\quad \dot\psi=\Omega_0,\quad \Omega_{z'}=\omega \quad (5)$$

if the constants θ_0, r_0, ϕ_0, Ω_0, ω satisfy the conditions

$$\frac{\partial}{\partial\theta}(T+U)=0,\quad \frac{\partial}{\partial r}(T+U)=0,\quad \frac{\partial}{\partial\varphi}(T+U)=0,\quad \frac{\partial}{\partial\delta}(T+U)=0$$

These conditions in our case have the form

$$-m\Omega_0^2(r_0^2\sin 2\theta_0\sin^2\varphi_0+2r_0\zeta\cos 2\theta_0\sin\varphi_0-\zeta^2\sin 2\theta_0)+$$
$$+(B_1-C_1)\Omega_0^2\sin 2\theta_0+2A\Omega_0^2(\sin\theta_0\cos\theta_0\cos^2\varphi_0+h_1h_3)-$$
$$-2C\omega\Omega_0h_4-2mg(r_0\cos\theta_0\sin\varphi_0-\zeta\sin\theta_0)=0 \quad (6)$$
$$2r_0\Omega_0^2(\cos^2\varphi_0+\sin^2\varphi_0\cos^2\theta_0)-\zeta\Omega_0^2\sin 2\theta_0\sin\varphi_0-2\mu_1r_0-2g\sin\theta_0\sin\varphi_0=0$$
$$mr_0\Omega_0^2(r_0\sin^2\theta_0\sin 2\varphi_0+\zeta\sin 2\theta_0\cos\varphi_0)+A\Omega_0^2(\sin^2\theta_0\sin 2\varphi_0-$$
$$-2h_3\sin\theta_0\cos\varphi_0\cos\delta_0)+2C\omega\Omega_0\sin\theta_0\cos\varphi_0\sin\delta_0+2mgr_0\sin\theta_0\cos\varphi_0=0$$
$$(Ah_2\Omega_0+C\omega)h_3\Omega_0+m\mu_2\delta_0=0$$

Here

$$h_1=\cos\theta_0\sin\varphi_0\cos\delta_0-\sin\theta_0\sin\delta_0,\qquad h_3=\sin\theta_0\sin\varphi_0\cos\delta_0+\cos\theta_0\sin\delta_0$$
$$h_2=\sin\theta_0\sin\varphi_0\sin\delta_0-\cos\theta_0\cos\delta_0,\qquad h_4=\cos\theta_0\sin\varphi_0\sin\delta_0+\sin\theta_0\cos\delta_0$$

We shall investigate the stability of the considered motion with respect to θ, r, ϕ, δ, $\dot\theta$, $\dot\psi$, $\dot r$, $\dot\phi$, $\dot\delta$, $\Omega_{z'}$. The perturbed motion will be denoted by

$$\theta=\theta_0+\eta_1,\quad r=r_0+\eta_3,\quad \varphi=\varphi_0+\eta_4,\quad \delta=\delta_0+\eta_5$$
$$\dot\theta=\xi_1,\quad \dot\psi=\Omega_0+\xi_2,\quad \dot r=\xi_3,\quad \dot\varphi=\xi_4,\quad \dot\delta=\xi_5,\quad \Omega_{z'}=\omega+\xi_6$$

The integrals of the perturbed motion corresponding to the integrals (2), (3), (4) are, respectively, V_1, V_2, V_3, where $V_3=\xi_6$. These integrals written in powers of $\eta_i (i=1,3,4,5)$, $\xi_i (i=1,\ldots,6)$ contain terms of the first and of the second order of magnitude only (terms of higher order are neglected).

It is easy to show, on the strength of (6), that the linear combination

$$W = V_1 - 2\Omega_0 V_2 - 2C(\omega + \Omega_0 h_2) V_3$$

does not contain the linear terms

$$W = F_1(\xi_1, \ldots, \xi_6) + F_2(\eta_1, \eta_3, \eta_4, \eta_5) + F_3(\eta_1, \eta_4, \eta_5, \xi_6) + \ldots$$

The quadratic form $F_1(\xi_1, \ldots, \xi_6)$ is positive-definite with respect to all its variables, because its determinant and the determinant of twice the kinetic energy expression, with values of coordinates as in (5), equal each other.

The functions F_2 and F_3 are also quadratic forms

$$\begin{aligned} F_2(\eta_1, \eta_3, \eta_4, \eta_5) = {} & [mr_0\Omega_0^2 (r_0 \cos 2\theta_0 \sin^2\varphi_0 + 2\zeta \sin 2\theta_0 \sin\varphi_0) - \\ & - (m\zeta^2 + A\cos^2\varphi_0 + B_1 - C_1)\Omega_0^2 \cos 2\theta_0 - A\Omega_0^2 (h_1^2 - h_3^2) - C\Omega_0\omega h_2 - \\ & - mg(r_0 \sin\theta_0 \sin\varphi_0 + \zeta\cos\theta_0)]\eta_1^2 + 2m\Omega_0^2 (r_0 \sin 2\theta_0 \sin^2\varphi_0 + \\ & + \zeta\cos 2\theta_0 \sin\varphi_0)\eta_1\eta_3 + [mr_0\Omega_0^2 (r_0 \sin 2\theta_0 \sin 2\varphi_0 + 2\zeta\cos 2\theta_0 \cos\varphi_0) + \\ & + A\Omega_0^2 (\sin 2\theta_0 \sin 2\varphi_0 - 2h_1 \sin\theta_0 \cos\varphi_0 \cos\delta_0 - 2h_3 \cos\theta_0 \cos\varphi_0 \cos\delta_0) + \\ & + 2C\omega\Omega_0 \cos\theta_0 \cos\varphi_0 \sin\delta_0 + 2mgr_0 \cos\theta_0 \cos\varphi_0]\eta_1\eta_4 + \\ & + 2\Omega_0 [A\Omega_0 (h_1 h_2 + h_3 h_4) + C\omega h_1]\eta_1\eta_5 + m[\mu_1 - \Omega_0^2 (\cos^2\varphi_0 + \\ & + \cos^2\theta_0 \sin^2\varphi_0)]\eta_3^2 + m\Omega_0^2 (2r_0 \sin^2\theta_0 \sin 2\varphi_0 + \zeta\sin 2\theta_0 \cos\varphi_0)\eta_3\eta_4 + \\ & + [mr_0\Omega_0^2 (r_0 \sin^2\theta_0 \cos 2\varphi_0 - {}^1/_2\zeta \sin 2\theta_0 \sin\varphi_0) + \\ & + A\Omega_0^2 (\sin^2\theta_0 \cos 2\varphi_0 + h_3 \sin\theta_0 \sin\varphi_0 \cos\delta_0 - \sin^2\theta_0 \cos^2\varphi_0 \cos^2\delta_0) - \\ & - C\omega\Omega_0 \sin\theta_0 \sin\varphi_0 \sin\delta_0]\eta_4^2 + 2[A\Omega_0^2 \sin\theta_0 \cos\varphi_0 (h_2 \cos\delta_0 + \\ & + h_3 \sin\delta_0) - C\omega\Omega_0 \sin\theta_0 \cos\varphi_0 \cos\delta_0]\eta_4\eta_5 + [m\mu_2 + A\Omega_0^2 (h_3^2 - h_2^2) - C\omega\Omega_0 h_2]\eta_5^2 \end{aligned}$$

$$F_3(\eta_1, \eta_4, \eta_5, \xi_6) = 2C\Omega_0 (h_4\eta_1 + \sin\theta_0 \cos\varphi_0 \sin\delta_0 \eta_4 + h_3\eta_5)\xi_6$$

If the function F_2 happens to be positive-definite with respect to the variables η_1, η_3, η_4, η_5, then it is easy to prove that the form

$$V = W + R\xi_6$$

can be made positive-definite with respect to all the perturbed coordinates and to all the velocities by selecting appropriate values for the constant R. Consequently, the form V can be regarded as the Liapunov function which solves the stability problem for solutions (5) [2]. In this way the sufficient condition for stability of the investigated motion with respect to θ, r, ϕ, δ, $\dot{\theta}$, $\dot{\psi}$, $\dot{r}$, $\dot{\phi}$, $\dot{\delta}$, $\Omega_{z'}$, is reduced to four conditions for positive-definiteness of the quadratic form $F_2(\eta_1, \eta_3, \eta_4, \eta_5)$ (the inequalities of Sylvester). In general, these four conditions are very complicated and involved. Let us consider certain special cases.

1. The rotation of the rotor in vertical position, $\theta_0 = 0$, $\psi_0 = 0$, $\phi_0 = 0$, $\delta_0 = 0$, is stable with respect to θ, r, δ and all the velocities

if the following inequalities

$$-(m\zeta^2 + A + B_1 - C_1)\,\Omega_0^2 + C\omega\Omega_0 - mg\zeta > 0 \qquad (7)$$
$$\mu_1 > \Omega_0^2, \qquad m\mu_2 + C\omega\Omega_0 - A\Omega_0^2 > 0$$

are satisfied.

2. The regular precession $\theta_0 \neq 0$, $r_0 = 0$, $\phi_0 = 0$, $\delta_0 = 0$ is stable with respect to the same variables if the following inequalities

$$-(m\zeta^2 + A + B_1 - C_1)\,\Omega_0^2 \cos 2\theta_0 + (C\ \Omega_0 - mg\zeta)\cos\theta_0 > 0 \qquad (8)$$
$$\mu_1 > \Omega_0^2, \qquad m\mu_2 + C\omega\Omega_0 \cos\theta_0 - A\Omega_0^2 \cos^2\theta_0 > 0 \qquad (9)$$

are satisfied.

The condition (8) is also the sufficient condition for stability of regular precession when the rotor axis is assumed to be rigid [3]. If the elastic properties of the rotor axis are taken into account, we need two additional inequalities as shown in (9).

The sufficient conditions for the stability of the solution (5) can be obtained from the Routh theorem.

The variable potential energy of the system has the form

$$\Pi = \frac{[P_\psi + P_\chi(\sin\theta\sin\varphi\sin\delta - \cos\theta\cos\delta)]^2}{n} + \frac{P_\chi^2}{C} +$$
$$+ m\mu_1 r^2 + m\mu_2\delta^2 + 2mg\,(r\sin\theta\sin\varphi + \zeta\cos\theta)$$

where

$$P_\psi = \partial T/\partial\dot\psi, \qquad P_\chi = \partial T/\partial\dot\chi$$
$$n = mr^2(\cos^2\varphi + \sin^2\varphi\cos^2\theta) - mr\zeta\sin 2\theta\sin\varphi + m\zeta^2\sin^2\theta +$$
$$+ B_1\sin^2\theta + C_1\cos^2\theta + I + A\sin^2\theta\cos^2\varphi + A(\sin\theta\sin\varphi\cos\delta + \cos\theta\sin\delta)^2$$

The investigated steady solution (5) is determined by the equations

$$\frac{\partial\Pi}{\partial\theta} = 0, \qquad \frac{\partial\Pi}{\partial r} = 0, \qquad \frac{\partial\Pi}{\partial\varphi} = 0, \qquad \frac{\partial\Pi}{\partial\delta} = 0 \qquad (10)$$

The solution (5) is stable with respect to θ, r, ϕ, δ, $\dot\theta$, $\dot r$, $\dot\phi$, $\dot\delta$, if in position (10) the function π has a minimum, and with the restriction that the constants of the cyclic integrals P_ψ, P_χ are not permitted to vary. The conditions for a minimum of π are given by four inequalities which are derived from the condition that in position (10) the principal diagonal minors of the determinant

$$\left\| \frac{\partial^2\Pi}{\partial q_i \partial q_j} \right\| \qquad \begin{matrix}(i, j = 1, \ldots, 4)\\ (q_1 = \theta,\ \ q_2 = r,\ \ q_3 = \varphi,\ \ q_4 = \delta)\end{matrix}$$

ought to be positive-definite.

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